CHAPTER 7

Olinde Rodrigues and Combinatorics

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Introduction

After more than twenty years of absence from mathematics, Olinde Rodrigues returned to publishing mathematical papers in 1838, as discussed in Chapter 2, with four short notes on combinatorial problems (Rodrigues 1838*a*, 1838*b*, 1838*c*, 1839).

Rodrigues's contributions of 1838 were part of a discussion in the pages of the *Journal de Liouville* that Olry Terquem (1782–1862) had initiated. Euler had provided a recursion for the number of different ways in which a polygon may be partitioned into triangles, but he never published a proof (although he evidently had one), and neither did later authors manage to provide one. Terquem informed Liouville of this fact, and he later communicated the problem to several other geometers. It was Rodrigues who, by careful analysis of Lamé's first proof from 1838, gave a direct derivation of the Euler identity. He found an even simpler proof for the equivalent problem of determining the number of different parenthesizations of a nonassociative expression, which had been addressed shortly before by Catalan. We shall take a closer look at the discussion in the *Journal de Liouville* in order to emphasize Rodrigues's contributions and to demonstrate that Catalan's part in this subject was not that important.

Rodrigues worked on a second combinatorial problem in his 1839 paper, also first discussed in the *Journal de Liouville* by Terquem. Here, he gave the generating function for the enumeration of permutations with a given number of inversions. We shall also briefly discuss another result of Rodrigues relevant to combinatorial theory, which is contained in his later paper (Rodrigues 1843), where he derived an approximation for the central binomial coefficient $\binom{2n}{n}$. Furthermore, we shall mention an application of Rodrigues's formula for orthogonal polynomials in the enumeration of alternating-sign matrices. We include this topic, because this application was so important that Bressoud (Bressoud 1999) devoted a whole section to Rodrigues in his book.

Finally, this survey will conclude with some remarks on Rodrigues's way of approaching combinatorial problems, on the role of the book by Netto (Netto 1901) in the propagation of Rodrigues's results, and on the fact that the problems that he considered are still under discussion today.

Catalan numbers

The Catalan numbers are the solution to a variety of enumeration problems of which Stanley (Stanley 1999) lists more that seventy works. For $n = 0, 1, 2, \dots$ the

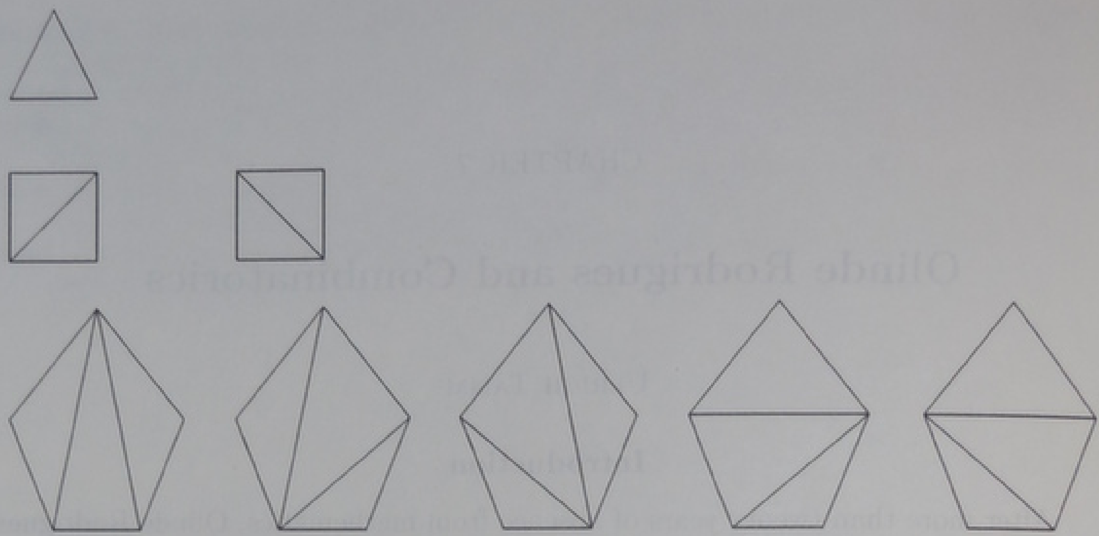

FIGURE 1. Partition of $(n+2)$-gons into triangles by nonintersecting diagonals.

nth *Catalan number* C_n is defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

Historically the first such problems were the polygon problem and the parenthesization problem, which were first discussed by Euler and Catalan, respectively.

Polygon Problem (Euler 1751): The Catalan number C_n, for $n = 1, 2, \ldots$, is the number of different partitions of an $(n+2)$-gon into triangles by means of nonintersecting diagonals (see Figure 1).

Parenthesization Problem (Catalan 1838): The Catalan number C_n, for $n = 1, 2, \ldots$, is the number of different parenthesizations of a nonassociative expression, where an expression in $n+1$ variables (in a fixed order) is evaluated by successive application of a binary operation $\circ$, e.g.,

$$\begin{aligned}
n = 1 &: (a \circ b),\\
n = 2 &: ((a \circ b) \circ c), (a \circ (b \circ c)),\\
n = 3 &: (((a \circ b) \circ c) \circ d), ((a \circ (b \circ c)) \circ d), ((a \circ b) \circ (c \circ d)),\\
&\quad (a \circ ((b \circ c) \circ d)), (a \circ (b \circ (c \circ d))).
\end{aligned}$$

Further problems, more important for current applications, in which the Catalan numbers occur as a counting function, have been considered in the nineteenth century by Cayley (Cayley 1859)—enumeration of binary, rooted trees—and by Whitworth (Whitworth 1879) and Bertrand (Bertrand 1878*a*)—enumeration of sequences under priority conditions, and the ballot problem.

Most of these problems can be extended in a natural way to enumerate similar configurations by the nth *generalized Catalan number* $C_n^{(m)}$, for a positive integer $m \geq 3$, defined as

$$C_n^{(m)} = \frac{1}{(m-1)n+1}\binom{mn}{n}. \tag{2}$$

One might, for instance, consider partitions of polygons into $(m+1)$-gons, expressions with m variables in one pair of parentheses, m-ary trees, etc.

The notion of Catalan numbers is standard nowadays in combinatorics. Catalan gave the explicit expression (1), which is implicit in Euler's recursion (3) below. In other branches of mathematics, they might also be known under different names. As Paulo Almeida pointed out to the author, rooted trees in quantum field theory are counted by CM-numbers (Connes-Moscovici). The notion of generalized Catalan numbers is not standard. Graham, Knuth, and Patashnik (Graham, Knuth, and Patashnik 1988, pp. 344–350) suggest that they be called Fuss numbers, but, perhaps, Binet numbers would be more appropriate, as we shall see later on.

The discussion in the *Journal de Liouville*: 1838–1843

For the early history of the Catalan numbers we refer to (Larcombe and Wilson 1998), where the presentation of Brown (Brown 1965) is corrected in some points. Seemingly, the problem of partitioning an n-gon into triangles was already discussed in China around 1730. Developments in Europe started in 1751 with a letter from Euler to Goldbach. Euler (Euler 1751) gave results for polygons of $3, \dots, 10$ sides using the formula $P_n = \frac{2\cdot 6\cdot 10\cdots 2(2n-5)}{2\cdot 3\cdot 4\cdots(n-1)}$ which is based on the recursion

$$P_{n+1} = \frac{4n-6}{n} P_n, \tag{3}$$

where P_n denotes the number of different partitions. He did not provide a derivation but stated that arriving at the proof had involved some considerable effort.

Euler also communicated the problem to von Segner, who produced a second recursion in (von Segner 1758):

$$P_{n+1} = P_n + P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3 + P_n. \tag{4}$$

Von Segner did not derive a formula for the numbers P_n from this recursion but only calculated the first values. A propagation error at P_{15} was, in turn, corrected by Euler (Euler 1758) on using (3). About thirty years later, Fuss (Fuss 1795), who received the problem from Pfaff, considered the more general problem of partitioning an n-gon into m-gons for a given m. He gave a recursion similar to von Segner's but did not find a formula for the numbers in question (in fact, these are just the generalized Catalan numbers).

This was state of the art in 1838, when the discussion in the *Journal de Liouville* began. We shall now go into this discussion more closely. In his biography of Liouville, Lützen (Lützen 1990, p. 102) states that '... Liouville did not just sit and wait for papers to be sent to the *Journal*. He often encouraged his colleagues and students to write papers, and he was often inspired to pursue ideas in the papers sent to the *Journal*.' For example, in 1839, after Liouville had told Jacques Binet about Rodrigues's work (Rodrigues 1838*a*) on the number of ways one can divide a polygon, Binet wrote a paper on the same subject (Binet 1839), which inspired Catalan (Catalan 1838) and Lamé (Lamé 1838) to similar studies, and a few years later Liouville (Liouville 1843) contributed to the discussion with a critical mention of a paper by Nicolas Fuss. In turn Binet (Binet 1843) commented on this in a note.

The problem was, in fact, brought to Liouville's attention by Terquem, who apparently had already found a solution, as stated in Liouville's footnote to Lamé (Lamé 1838): '... M. Terquem having succeeded with the help of certain properties of factorial has proposed this problem to me. I have consequently communicated it to several geometers, no one of them has solved it; M. Lamé has been more fortunate, I ignore if others had obtained before him such an elegant solution.' So it was Lamé (Lamé 1838) who published the first proof that the numbers obtained via Euler's recursion (3) may be derived from von Segner's recursion (4). His idea was to use a second recursion (the background is presented in the next section), namely

$$P_n = \frac{n}{2n-6}(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3), \tag{5}$$

which then is simply plugged into (4) in order to obtain the recursion (3).

Catalan's first paper on the subject was published next to Lamé's article. It does not contain a proof but introduces the parenthesization problem and shows that it must have the same solution, because the solutions must obey the same recursion (4). Furthermore, Catalan derives the closed expression (1) from Euler's recursion (3), which is, of course, immediate.

In the same year 1838, Rodrigues published three small notes. In (Rodrigues 1838*a*), he gave a direct combinatorial proof (avoiding von Segner's recursion) for the Euler identity (4) via the polygon problem. This proof is based on Lamé's ideas. An even simpler proof, via the parenthesization problem, is presented in (Rodrigues 1838*b*), from which (4) can easily be derived. The third note, (Rodrigues 1838*c*), is a reflection on the binomial series, which comes into play in Binet's analytical derivation of (4). Binet (Binet 1839) obtained analytically the numbers (1) and hence the recursion (3) from von Segner's recursion (4). Note that $C_n = P_{n+2}$ for $n \geq 1$, so from von Segner's recursion (4) it is clear that the generating function

$$u(x) = C_0 + C_1x + C_2x^2 + \ldots$$

satisfies the functional equation

$$u(x) = 1 + xu(x)^2.$$

This equation has the root $u(x) = \frac{1-\sqrt{1-4x}}{2x}$. The coefficients of the binomial series, applied to $(1-4x)^{1/2}$, finally yield the numbers (1).

Further papers on the subject were published in 1839 by Catalan (Catalan 1839*a*, 1839*b*) and by Duhamel (Duhamel 1839) in the *Journal de Liouville*. In the first issue of the journal *Archiv der Mathematik und Physik* its editor, Grunert (Grunert 1841), wanted to draw the attention of the above-mentioned French mathematicians to the paper of Fuss (Fuss 1795), who had considered the more general problem on the partition of an n-gon into m-gons: 'The first purpose of this paper is to draw the attention of the above-mentioned gentlemen to the fact, which seems completely missed by them, that already an older excellent mathematician Nicolaus Fuss, in *Novis Actis Academiae scientiarum imperialis Petropolitanae*, T. IX, p. 243 had made a start with a much more general problem presented to him, according to his own words, by Johann Friedrich Pfaff, namely the problem to determine the number of different ways in which an n-gon can be subdivided into m-gons by diagonals.' So Grunert suggested that the French mathematicians were totally unaware of the Fuss paper and that Fuss had made a start (ausgelöst) on a more general problem.

Two years later, in (Liouville 1843), this author made some remarks on Fuss's paper. This should be taken as a reply to Grunert, but Liouville did not mention his name at all. Perhaps the word 'ausgelöst' in the Fuss paper was interpreted as 'gelöst', which means solved, since Liouville pointed out quite sharply that Fuss had not solved the problem but in fact had missed a solution at several points. Liouville admitted that Fuss had already found a recursion, from which Lamé's recursion (5) would follow for the special case of the triangles, but he stated that Fuss had not compared his results with von Segner's recursion (4) and hence had missed Lamé's proof. Furthermore, Liouville wrote that Fuss had been aware of the functional equation

$$u(x) = 1 + xu(x)^{m-1}. \tag{6}$$

For $m = 3$, this equation was the central tool in Binet's analytical derivation of (3), which again had been missed by Fuss. Finally, Fuss did not realize either that even for the more general problem, a solution can be obtained from (6) by Lagrange inversion. This calculation was finally carried out by Binet (Binet 1843) in the note following that of Liouville (Liouville 1843). Binet derived the numbers in (2) via the Lagrange inversion as a solution to the problem discussed by Fuss. The Lagrange inversion turned out to be an important tool in algebraic enumeration (cf., e.g., (Gessel and Stanley 1996)).

Olinde Rodrigues's contributions to Catalan numbers

In order to understand the contribution to the subject given by (Rodrigues 1838a), we must briefly review the recursions (4) and (5).

Von Segner obtained his recursion (4) by using the following argument. Consider the polygon $(1, 2, 3, \ldots, n+1)$ on $n+1$ vertices, where sides of the polygon connect two vertices i and j, when $|i-j| = 1 \bmod n+1$. Now fix one side, $(1, n+1)$ say, and notice that the triangle $(1, n+1, k)$ subdivides the original polygon into two smaller polygons of size k and $n+2-k$, respectively, which can be partitioned into triangles in P_k and P_{n-k+2} ways. Since a triangle containing the side $(1, n+1)$ must be contained in every partition, on repeating this subdivision with all triangles $(1, n+1, k)$, $k = 2, 3, \ldots, n$, all partitions of the original polygon are obtained. Lamé and Fuss's idea behind the second recursion (5) is very similar: They fix a vertex, 1 say, and consider all diagonals $(1, k)$ that subdivide the original polygon $(1, 2, \ldots, n)$ on n vertices into two smaller polygons of size k and size $n-k+2$. Since not all partitions can be obtained this way, the procedure is repeated by fixing subsequently the vertices $i = 2, 3, \ldots, n$ and then subdividing the polygon with all diagonals (i, k). In this way, the polygon $(1, 2, \ldots, n)$ is partitioned into triangles n times and each triangle occurs in exactly $2n-6$ of the partitions, yielding the recursion (5).

Rodrigues (Rodrigues 1838a) observed that these two ideas may be combined to obtain a direct derivation of the recursion (3). He simply identified the two vertices 1 and $n+1$ of the $(n+1)$-gon in the first construction to obtain a single vertex, 1 say, in an n-gon. Of course, accordingly, the triangles collapse into diagonals. Careful enumeration gives the recursion (3) without using (4) and (5).

In his second paper on the subject, Rodrigues (Rodrigues 1838b) provided a second, even shorter, proof of (3) via the parenthesization problem. Notice that in the original problem the variables in the expression are in fixed order: $a_1 \circ a_2 \circ \cdots \circ a_n$, say. Rodrigues in fact derived a recursion for the total number B_n of all parenthesizations of expressions, where the variables $a_1, \ldots, a_n$ may occur in any order. Since there are $n!$ many permutations of the set of variables, finally one has to divide by $n!$ to obtain the Catalan numbers. Now assume that a new variable

a_{n+1} is inserted into an expression E, with a well-formed parenthesization of the variables $a_1, \ldots, a_n$. Either a_{n+1} is placed at the beginning or at the end, yielding the two expressions $(a_{n+1} \circ (E))$ and $((E) \circ a_{n+1})$, or a_{n+1} is inserted into one of the $n-1$ pairs of parentheses, which can be done in four ways for each such pair. To see this, assume that the parentheses combine the expressions C and D in E. Then the four possibilities for inserting a_{n+1} into $(C \circ D)$ are

$$((a_{n+1} \circ C) \circ D), ((C \circ a_{n+1}) \circ D), (C \circ (a_{n+1} \circ D)), (C \circ (D \circ a_{n+1})).$$

So, $B_{n+1} = 2B_n + 4(n-1)B_n = (4n-2)B_n$. By the previous considerations $P_{n+1} = \frac{B_n}{n!}$. The proof can be presented in an even shorter, different way, which Olinde Rodrigues probably had in mind. Each expression with n variables (in arbitrary order) may be regarded as a string of $4n-3$ symbols, namely the n variables $a_1, \ldots, a_n$, $n-1$ operators $\circ$, and $2n-2$ parentheses. Now, with a new variable a_{n+1}, a new operator $\circ$, and a new pair of parentheses must also be inserted. For the position of a_{n+1} there are exactly $4n-2$ choices, before or behind the previous expression or between any two symbols in the previous expression. The positions of the other new symbols $\circ$, (, and) to be inserted are then automatically determined.

In his third note from the same year, Rodrigues (Rodrigues 1838*c*) provided an algebraic and elementary derivation—without using the Taylor series—for the expansion of the binomial series $(1+x)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} x^j$ when α is a negative integer or a rational number. This expansion for $\alpha = \frac{1}{2}$ was the central tool in Binet's analytical proof of Euler's recursion (3).

Netto's presentation

Netto (Netto 1901) devoted considerable space in his book to Catalan numbers without referring to them as such—(Netto, 1901, Sections 122–125) this author devoted Sections 122–125, Chapter 9, to Catalan numbers (but he did not use this notion yet). In Section 122 he first introduced the parenthesization problem, correctly stating in a footnote that it had first been treated by Catalan. He continued with Binet's analytical derivation of the Euler recursion (without mentioning Binet or Euler), and in Section 123 he gave Rodrigues's direct proof (referring to Rodrigues). In Section 124 he pointed out the connection to the problem of partitioning a polygon into triangles. His formulation, however, is extremely unhelpful: 'E. Catalan pointed out that the problem under discussion is connected to the geometrical question: in how many ways can one subdivide a polygon in the plane by diagonals?' To a reader, not familiar with the complete story, his presentation may suggest that, chronologically, the parenthesization problem was discussed before the polygon problem—especially since no further reference to Euler or Lamé (whose paper came before Catalan's) is given—and that everything started with Catalan. It is not clear why Netto presented the topic in this way. Perhaps the parenthesization problem was just more suitable for him because the rest of the chapter is devoted to similar enumeration problems due to Schröder or perhaps because he wanted to include a direct derivation and preferred Rodrigues's proof of the parenthesization problem to the one for the polygon partitioning problem. In any case his book was actually the first textbook on combinatorics (a previous book by Hindenburg published around 1800 did not circulate widely), and his presentation might have been the source from which misconceptions have originated.

Sequences and inversions

In his fourth paper, (Rodrigues 1839), published a year later than the other three, Rodrigues discussed the problem of counting the number of permutations of n elements with exactly i inversions. Indicating this number by $Z_{n,i}$, he showed that $Z_{n,i}$ is the coefficient of t^i in the generating function

$$(1+t)(1+t+t^2)(1+t+t^2+t^3)\cdots(1+t+t^2+\cdots+t^{n-1}) = \frac{\prod_{j=1}^{n}(1-t^j)}{(1-t)^n}.$$

On taking the derivative of this generating function, he obtained a new solution for a problem posed by Stern and later proved by Terquem, namely how to determine the sum of all the inversions of the $n!$ permutations on $\{1,2,\ldots,n\}$, which is $\frac{n!}{2}\cdot\frac{n(n-1)}{2}$. Finally, he further simplified the analysis of the numbers $Z_{n,i}$ by demonstrating that they can be written as

$$Z_{n,i} = \binom{n+i-1}{i} + \binom{n+i-2}{i-1}E_{1,1} + \binom{n+i-3}{i-2}E_{2,2} + \cdots + E_{i,i},$$

where $E_{n,i}$ is the coefficient of t^i in the product $\prod_{j=1}^{n}(1-t^j)$.

Netto (Netto 1901) included a chapter on sequences and inversions in his book without mentioning Rodrigues's results. Carlitz (Carlitz 1970), who derived the generating function for the more general problem of enumerating sequences with k_1 1's, k_2 2's , ..., k_n n's according to a given number of inversions, redressed this failure, C. A. Church having informed him of Rodrigues's work. Of course, for the special case $k_1 = k_2 = \cdots = k_n = 1$, the permutations discussed by Rodrigues arise. For further applications, Chapter 6 should be consulted.

Approximation of central binomial coefficients

It is important for many applications to have a good approximation of binomial or even multinomial coefficients. For instance, the entropy function of thermodynamics and information theory, defined as $H(p_1,\ldots,p_m) = -\sum_{i=1}^{m} p_i \cdot \log p_i$, for $p_1,\ldots,p_m \geq 0$ with $\sum_{i=1}^{m} p_i = 1$, may be obtained by approximation of the multinomial coefficients $\binom{n}{(np_1)\cdots(np_m)} \approx 2^{n\cdot H(p_1,\ldots,p_m)}$. Furthermore, the limit theorem of De Moivre and Laplace is derived by an appropriate approximation of the binomial coefficients. Usually, Stirling's formula is applied to the single factorials.

For various applications in probability theory an approximation of the central binomial coefficients $\binom{2n}{n}$ is of special importance. This was extensively discussed by Laplace (Laplace 1814) in his *Théorie Analytique des Probabilités.* A later application is the demonstration of the nonrecurrence of the random walk on the line, where the Catalan numbers come into play in the formula for the expected value.

Rodrigues (Rodrigues 1843) provides a new method for the approximation of $\binom{2n}{n}$ based on the development of trigonometric functions and on the Wallis product $\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n)^2-1}$. As he points out, his method yields an approximation of the same order as the one resulting from the first three terms of the approximation presented by Laplace (who used the coefficients occurring in Stirling's formula) and additionally gives bounds for the error involved.

The Rodrigues formula and alternating-sign matrices

The last work of Olinde Rodrigues concerned with combinatorics that we shall discuss in this paper is not an original result of his, but it is an application of the formula for orthogonal polynomials still known by his name today. However, we include it here since it is quite unusual to find a whole section of a mathematics book devoted to Olinde Rodrigues, as is the case in the book by Bressoud (Bressoud 1999) on alternating-sign matrices. An alternating-sign matrix is a square matrix with entries from $\{0, 1, -1\}$ such that (i) the entries in each row and column sum up to 1 and (ii) the nonzero entries in each row and column alternate in sign. An example is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

These matrices were discovered in the analysis of an algorithm to evaluate determinants invented by Charles Dodgson, better known as Lewis Carroll. There are also deep relations with problems in statistical mechanics, such as the configuration of water molecules in 'square ice', which can be described by an alternating-sign matrix.

The alternating-sign matrix conjecture concerns the total number of $n \times n$ alternating-sign matrices, which was conjectured to be $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. The problem was open for fifteen years until it was finally settled by Zeilberger (Zeilberger 1996). A discrete version of the Legendre polynomials comes in as an important step in the proof. Bressoud (Bressoud 1999, p. 252) points out in his section on Rodrigues that '[t]his work would be naught unless we had an alternate description of the polynomials $P_n(x)$. In 1816 Olinde Rodrigues observed that the nth Legendre Polynomial could be written as $\frac{n!}{(2n)!} D^n[(x^2-1)^n]$.'

Mathematicians are still looking for a simpler and more natural derivation of the formula for the total number of alternating-sign matrices. This formula also arises as a Hankel determinant where the entries in the matrix are the coefficients of the generating function $\frac{1-(1-9x)^{1/3}}{3x}$, the binomial series studied by Rodrigues (Rodrigues 1838c) with exponent $\frac{1}{3}$. An appropriate combinatorial interpretation of these numbers might yield a new proof of the alternating-sign matrix conjecture. An idea would be to look for an explanation of the recursion $P_{n+1} = \frac{6n-4}{n} P_n$ in the spirit of Rodrigues (Rodrigues 1838b) and then to find a bijection to a problem concerning disjoint lattice paths, which are enumerated by Hankel determinants.

Concluding remarks

Olinde Rodrigues's approach to combinatorial problems led to very elegant solutions. This holds for the direct proofs for the polygon and parenthesization problems as well as for the derivation of the generating function for the numbers of permutations according to their inversions. His constant search for simplicity in his proofs is striking. Some quotations from his papers may support this impression: '...formula that M. Laméhas proved in the first issue of this Journal and that can be established more directly below' (Rodrigues 1838a, p. 547). It is clear that

Rodrigues looks whenever it is possible for elementary proofs: '[I] give here the direct and elementary solution' (Rodrigues 1838*b*, p. 549); '[e]lementary and purely algebraic proof...' (Rodrigues 1838*c*). Also, simplicity is never far from his mind: '[o]ne can, however, arrive, more simply at this result' (Rodrigues 1839, p. 238) or '[t]his is the simplest expression which I have found' (Rodrigues 1839, p. 239).

The two combinatorial problems on which Rodrigues worked in 1838 and 1839, were brought to the interest of French mathematicians by Olry Terquem, although little is known at present about the connection between Rodrigues and Terquem.

Contrasting with almost all his other results, Rodrigues's work on the Catalan numbers is still remembered nowadays. As mentioned before, Netto (Netto 1901) included the proof from (Rodrigues 1838*b*) in his book, and Rodrigues is also frequently cited in further publications on this topic. Strangely, Netto (Netto 1901) does not refer to Rodrigues's paper of 1839, although in the same book he also addresses the problem of enumerating permutations with a given number of inversions. Rodrigues's result on the generating function for these permutations was forgotten until Carlitz (Carlitz 1970) referred to it. Even nowadays it is not widely known that this generating function goes back to Rodrigues; in a recent paper on the subject Margolius ((Margolius 2001), for instance) attributed it to a work by Muir of 1898.

The reference of Margolius also demonstrates that the topics that Olinde Rodrigues considered are still under discussion today. This is even more obvious for the Catalan numbers, which are quite in fashion at the present time, their role in the enumeration of binary trees having made them of importance in computer science. Such trees may serve as a data structure, for instance in data compression (cf. (Kobayashi, Morita, and Hoshi 1996)). Knuth (Knuth 1973) addressed the problem of sorting a sequence through a stack, which is another application of the Catalan numbers. Generalizations to sorting problems with two or more stacks have recently been discussed so it is not surprising that some new books on discrete mathematics devote a chapter to Catalan numbers. For new combinatorial developments in this context the survey by Aigner (Aigner 2001) should be consulted, and their relation to orthogonal polynomials is also discussed in (Tamm 2001). In this way, Hankel matrices of generalized Catalan numbers are related to alternating-sign matrices. There is no better way to gauge the originality and importance of Rodrigues's combinatorial works than to quote a comment on Rodrigues's mathematical talent by Bertrand(Bertrand 1878*b*). 'A very elementary problem, apparently very simple, made it soon evident the flexibility of his talent. Segner had proposed it in the last century, and two very different formulae—the most elegant due to Euler—had necessarily to provide identical results; but this identity was not easy to establish [....] Olry Terquem has succeeded by a long and tortuous route; he proposed the problem to M. Liouville who communicated it to several geometers none of whom succeeded in solving it. Lamé was luckier and sent a solution the day after he had received the enunciation. The first place in this fight [...] was not, however, reserved for him; Lamé's elegant solution woke up an eminent spirit, for a long time oblivious to science. Olinde Rodrigues, in a note a few pages long, reminded his old fellow students, who by then had become great masters, that in the past, at the *lycée* he led them all and that, had he wished, he could have become their equal.'

Bibliography

Aigner, M. (2001). Catalan and other numbers: a recurrent theme. In *Algebraic combinatorics and computer science. A tribute to Gian-Carlo Rota* (H. Crapo and D. Senato, eds.), pp. 347–390. Springer-Verlag Italia, Milan.

Bertrand, J. (1878*a*). Solution d'un problème. *Comptes Rendus de l'Académie des Sciences*, Paris, **105**, 369.

Bertrand, J. (1878*b*). Élogue de Gabriel Lamé, ingénieur en chef the mines lu dans la scéance publique annuelle du 28 janvier 1878. *Annales de Mines*, **7**, 13.

Binet, J. (1839). Réflexions sur le problème de déterminer le nombre de manières dont une figure rectiligne peut être partagée en triangles au moyen de ses diagonales. *Journal de Mathématiques Pures et Appliquées*, **4**, 79–91.

Binet, J. (1843). Note de M. J. Binet. *Journal de Mathématiques Pures et Appliquées*, **8**, 394–396.

Bressoud, D. M. (1999). *Proofs and confirmations: The story of the alternating sign matrix conjecture*. Cambridge University Press.

Brown, W. G. (1965). Historical note on a recurrent combinatorial problem. *American Mathematical Monthly*, **72**, 973–977.

Carlitz, L. (1970). Sequences and inversions. *Duke Mathematical Journal*, **37**, 193–198.

Catalan, E. (1838). Note sur une équation aux différences finies. *Journal de Mathématiques Pures et Appliquées*, **3**, 508–516.

Catalan, E. (1839*a*). Solution nouvelle de cette question: un polygone étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales. *Journal de Mathématiques Pures et Appliquées*, **4**, 91–94.

Catalan, E. (1839*b*). Addition à la note sur une équation aux différences finies insérée dans le volume précédent, page 508. *Journal de Mathématiques Pures et Appliquées*, **4**, 95–99.

Cayley, A. (1859). On the analytical forms called trees—part II. *Philosophical Magazine, 4th series*, **18**, 374–378.

Duhamel, J. M. C. (1839). Intégration d'une équation aux différences. *Journal de Mathématiques Pures et Appliquées*, **4**, 222–224.

Euler, L. (1751). Letter to Christian Goldbach dated 4 September 1751.

Euler, L. (1758). Enumeratio modorum, quibus figurae planae rectilineae per diagonales dividintur in triangula, auctore J. A. de Segner, page 203. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, **7**, 13–15.

Fuss, N. (1795). Solutio quaestionis, quot modis polygonum n laterur in polygonam laterum, per diagonales resolvi queat. *Nova Acta Academiae Scientiarum Imperialis Petropolitanae*, **9**, 243–251.

Gessel, I. and Stanley, R. P. (1996). Algebraic enumeration. In *Handbook of combinatorics, Vol. 2* (R. L. Graham, M. Grötschel, and L. Lovasz, eds.), pp. 1021–1069, Elsevier Science B. V., Amsterdam.

Graham, R. L., D. E. Knuth, and O. Patashnik (1988). *Concrete mathematics*. Addison Wesley, Reading.

Grunert, J. A. (1841). Über die Bestimmung der Anzahl der verschiedenen Arten, auf welche sich ein $n$eck durch Diagonalen in lauter $m$ecke zerlegen lässt, mit Bezug auf einige Abhandlungen der Herren Lamé, Rodrigues, Binet, Catalan und Duhamel in dem Journal de Mathématiques pures et appliquées, publié par Joseph Liouville, Vols. 3, 4. *Archiv der Mathematik und Physik*, **1**, 193–203.

Knuth, D. E. (1973). *The art of computer programming*, Vol. 1, *Fundamental algorithms*. Addison Wesley, Reading.

Kobayashi, K., Morita, H., and Hoshi, M. (1996). Enumerative coding for k-ary trees. *Proceedings of the 19th Symposium on Information Theory and its Applications (SITA96)*, Hakone, Japan, pp. 377–379.

Lamé, G. (1838). Extrait d'une lettre de M. Lamé à M. Liouville sur cette question: un polygone convexe étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales? *Journal de Mathématiques Pures et Appliquées*, **3**, 505–507.

Laplace, P. S. (1814) *Théorie analytique des probabilités (2nd ed., rev. et augm. par l'auteur)*. Courcier, Paris.

Larcombe, P. J. and Wilson, P. D. C. (1998). On the trail of the Catalan sequence. *Mathematics Today*, **34**, 114–117. (Also *ibid*, **35**, 25 and 89).

Liouville, J. (1843). Remarques sur un mémoire de N. Fuss. *Journal de Mathématiques Pures et Appliquées*, **8**, 391–394.

Lützen, J. (1990). *Joseph Liouville 1809–1882: Master of pure and applied mathematics.* Studies in the History of Mathematics and Physical Sciences, Vol. 15. Springer-Verlag, New York.

Margolius, B. H. (2001). Permutations with inversions. *Journal of Integer Sequences*, **4**, Article 01.2.4, 13 pp. (electronic).

Netto, E. (1901). *Lehrbuch der Combinatorik.* Teubner, Leipzig.

Rodrigues, O. (1838*a*). Sur le nombre de manières de décomposer un polygone en triangles au moyen de diagonales. *Journal de Mathématiques Pures et Appliquées*, **4**, 547–548.

Rodrigues, O. (1838*b*). Sur le nombre de manières d'effectuer un produit de n facteurs. *Journal de Mathématiques Pures et Appliquées*, **4**, 549.

Rodrigues, O. (1838*c*). Démonstration élémentaire et purement algébrique du développement d'un binôme élevé à une puissance négative ou fractionnaire. *Journal de Mathématiques Pures et Appliquées*, **4**, 550–551.

Rodrigues, O. (1839). Note sur les inversions, ou dérangements produits dans les permutations. *Journal de Mathématiques Pures et Appliquées*, **5**, 236–240.

Rodrigues, O. (1843). Du développement des fonctions trigonométriques en produits de facteurs binômes. *Journal de Mathématiques Pures et Appliquées*, **8**, 217–224.

Stanley, R. P. (1999). *Enumerative combinatorics*, Vol. 2. Cambridge University Press.

von Segner, J. A. (1758). Enumeratio modorum, quibus figurae planae rectilineae per diagonales dividintur in triangula. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, **7**, 203–210.

Tamm, U. (2001). Some aspects of Hankel matrices in combinatorics and coding theory. *The Electronic Journal of Combinatorics*, **8**, # A1, 31 pp. (electronic).

Whitworth, M. A. (1879). Arrangements of m things of one sort and n things of another sort, under certain conditions of priority. *Messenger of Mathematics*, **8**, 105–114.

Zeilberger, D. (1996). Proof of the refined alternating sign matrix conjecture. *New York Journal of Mathematics*, **2**, 59–68.